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# New cohomogeneity one metrics with Spin(7) holonomy

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## Abstract

We construct new explicit non-singular metrics that are complete on non-compact Riemannian 8-manifolds with holonomy Spin(7). One such metric, which we denote by  $\mathbb{A}_8$ , is complete and non-singular on  $\mathbb{R}^8$ . The other complete metrics are defined on manifolds with the topology of the bundle of chiral spinors over  $S^4$ , and are denoted by  $\mathbb{B}_8^+$ ,  $\mathbb{B}_8^-$  and  $\mathbb{B}_8$ . The metrics on  $\mathbb{B}_8^+$  and  $\mathbb{B}_8^-$  occur in families with a non-trivial parameter. The metric on  $\mathbb{B}_8$  arises for a limiting value of this parameter, and locally this metric is the same as the one for  $\mathbb{A}_8$ . The new Spin(7) metrics are asymptotically locally conical (ALC): near infinity they approach a circle bundle with fibres of constant length over a cone whose base is the squashed Einstein metric on  $\mathbb{C}\mathbb{P}^3$ . We construct the covariantly constant spinor and calibrating 4-form. We also obtain an  $L^2$ -normalisable harmonic 4-form for the  $\mathbb{A}_8$  manifold, and two such 4-forms (of opposite dualities) for the  $\mathbb{B}_8$  manifold.

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## 1. Introduction

Few explicit examples of complete non-compact manifolds admitting Ricci-flat metrics with the exceptional holonomies  $G_2$  in seven dimensions or Spin(7) in eight dimensions are

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known. Three asymptotically conical examples have been found in  $D = 7$ , for manifolds with the topology of the bundle of self-dual 2-forms on  $S^4$  or  $\mathbb{C}\mathbb{P}^2$ , and the spin bundle of  $S^3$  [1,5]. In  $D = 8$  the only Spin(7) example that was known was defined on the chiral spin bundle of  $S^4$  [1,5].

In this paper, we give a construction of new eight-dimensional metrics of Spin(7) holonomy, and show that among these are examples that are complete on two different non-compact manifolds. It represents the elaboration and simplification of the original construction in [4]. The new metrics are all asymptotically locally conical (ALC), locally approaching  $\mathbb{R} \times S^1 \times \mathbb{C}\mathbb{P}^3$ . The radius of the  $S^1$  is asymptotically constant, so the metric approaches an  $S^1$  bundle over a cone with base  $\mathbb{C}\mathbb{P}^3$ . However, the Einstein metric on the  $\mathbb{C}\mathbb{P}^3$  at the base of the cone is not the Fubini–Study metric, but instead the “squashed” metric described as an  $S^2$  bundle over  $S^4$ . The new solutions can have very different short-distance behaviours, with one approaching flat  $\mathbb{R}^8$  whilst all the others approach  $\mathbb{R}^4 \times S^4$  locally. The global topology is that of  $\mathbb{R}^8$  in the first case and the bundle of positive (or negative) chirality spinors over  $S^4$  for the others. An intriguing feature of two of the new metrics, one on each of the inequivalent topologies, is that in local coordinates the metrics are identical. Globally, the metric is complete on a manifold of  $\mathbb{R}^8$  topology if the radial coordinate  $r$  is taken to be positive, whilst in the region with negative  $r$  it is instead complete on the manifold  $\mathbb{S}(S^4)$  of the bundle of chiral spinors over  $S^4$ . We shall denote the new Spin(7) manifold with  $\mathbb{R}^8$  topology by  $\mathbb{A}_8$ , and the new related manifold with  $\mathbb{S}(S^4)$  topology by  $\mathbb{B}_8$ . The more general classes of new manifolds with the topology of the chiral spin bundle over  $S^4$  will be denoted by  $\mathbb{B}_8^+$  and  $\mathbb{B}_8^-$ .

Our construction is a generalisation of the one that leads to the previously known metric of Spin(7) holonomy. That example is given, in local coordinates, by [1,5]:

$$ds_8^2 = \left[ 1 - \left( \frac{r_0}{r} \right)^{10/3} \right]^{-1} dr^2 + \frac{9}{100} r^2 \left[ 1 - \left( \frac{r_0}{r} \right)^{10/3} \right] h_i^2 + \frac{9}{20} r^2 d\Omega_4^2, \tag{1}$$

where

$$h_i \equiv \sigma_i - A^i, \tag{2}$$

the  $\sigma_i$  are left-invariant 1-forms on SU(2),  $d\Omega_4^2$  the metric on the unit 4-sphere, and  $A^i$  the potential of the BPST SU(2) Yang–Mills instanton on  $S^4$ . The  $\sigma_i$  can be written in terms of Euler angles as

$$\begin{aligned} \sigma_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\varphi, & \sigma_2 &= -\sin \psi d\theta + \cos \psi \sin \theta d\varphi, \\ \sigma_3 &= d\psi + \cos \theta d\varphi. \end{aligned} \tag{3}$$

The principal orbits are  $S^7$ , viewed as an  $S^3$  bundle over  $S^4$ . The solution (1) is asymptotic to a cone over the “squashed” Einstein 7-sphere, and it approaches  $\mathbb{R}^4 \times S^4$  locally at short distance (i.e.  $r \approx \ell$ ). Globally the manifold has the same topology  $\mathbb{S}(S^4)$ , the bundle of chiral spinors over  $S^4$ , as the new Spin(7) manifolds  $\mathbb{B}_8$  and  $\mathbb{B}_8^\pm$  that we obtain in this paper.

## 2. Einstein equation and first integrals for Spin(7) metrics

The generalisation that we shall consider involves allowing the  $S^3$  fibres of the previous construction themselves to be “squashed”. In particular, this encompasses the possibility

of having an asymptotic structure of the “Taub-NUT type,” in which the  $U(1)$  fibres in a description of  $S^3$  as a  $U(1)$  bundle over  $S^2$  approach constant length while the radius of the  $S^2$  grows linearly.

A convenient way to parameterise the metric is by first introducing the left-invariant 1-forms  $L_{AB}$  for the group manifold  $SO(5)$ . These satisfy  $L_{AB} = -L_{BA}$ , and

$$dL_{AB} = L_{AC} \wedge L_{CB}. \tag{4}$$

The 7-sphere is then given by the coset  $SO(5)/SU(2)_L$ , where we take the obvious  $SO(4)$  subgroup of  $SO(5)$ , and write it (locally) as  $SU(2)_L \times SU(2)_R$ . If we take the indices  $A$  and  $B$  in  $L_{AB}$  to range over the values  $0 \leq A \leq 4$ , and split them as  $A = (a, 4)$ , with  $0 \leq a \leq 3$ , then the  $SO(4)$  subgroup is given by  $L_{AB}$ . This is decomposed as  $SU(2)_L \times SU(2)_R$ , with the two sets of  $SU(2)$  1-forms given by the self-dual and anti-self-dual combinations:

$$R_i = \frac{1}{2}(L_{0i} + \frac{1}{2}\epsilon_{ijk}L_{jk}), \quad L_i = \frac{1}{2}(L_{0i} - \frac{1}{2}\epsilon_{ijk}L_{jk}), \tag{5}$$

where  $1 \leq i \leq 3$ . Thus the seven 1-forms in the  $S^7$  coset will be

$$R_1, R_2, R_3, P_a \equiv L_{a4}. \tag{6}$$

It is straightforward to establish that

$$\begin{aligned} dP_0 &= (R_1 + L_1) \wedge P_1 + (R_2 + L_2) \wedge P_2 + (R_3 + L_3) \wedge P_3, \\ dP_1 &= -(R_1 + L_1) \wedge P_0 - (R_2 - L_2) \wedge P_3 + (R_3 - L_3) \wedge P_2, \\ dP_2 &= (R_1 - L_1) \wedge P_3 - (R_2 + L_2) \wedge P_0 - (R_3 - L_3) \wedge P_1, \\ dP_3 &= -(R_1 - L_1) \wedge P_2 + (R_2 - L_2) \wedge P_1 - (R_3 + L_3) \wedge P_0, \\ dR_1 &= -2R_2 \wedge R_3 - \frac{1}{2}(P_0 \wedge P_1 + P_2 \wedge P_3), \\ dR_2 &= -2R_3 \wedge R_1 - \frac{1}{2}(P_0 \wedge P_2 + P_3 \wedge P_1), \\ dR_3 &= -2R_1 \wedge R_2 - \frac{1}{2}(P_0 \wedge P_3 + P_1 \wedge P_2). \end{aligned} \tag{7}$$

In terms of these left-invariant 1-forms, we can write the ansatz for the more general metrics of  $Spin(7)$  holonomy on the  $\mathbb{R}^4$  bundle over  $S^4$  as

$$ds_8^2 = dt^2 + 4a^2(R_1^2 + R_2^2) + 4b^2R_3^2 + c^2P_a^2. \tag{8}$$

We shall work with an orthonormal frame bundle  $e^A$  defined by

$$e^8 = dt, \quad e^{\hat{1}} = 2aR_1, \quad e^{\hat{2}} = 2aR_2, \quad e^{\hat{3}} = 2bR_3, \quad e^a = cP_a, \tag{9}$$

where we take the index  $a$  to range over  $0 \leq a \leq 3$ .

The factors of 4 in the terms involving  $a^2$  and  $b^2$  in (8) are included for consistency with the conventions in [4]. In that paper, the metric was written as

$$d\hat{s}_8^2 = dt^2 + a^2(D\mu^i)^2 + b^2\sigma^2 + c^2d\Omega_4^2, \tag{10}$$

where

$$D\mu^i \equiv d\mu^i + \epsilon_{ijk}A^j\mu^k, \quad \sigma \equiv d\varphi + \mathcal{A}, \quad \mathcal{A} \equiv \cos\theta d\psi - \mu^i A^i \tag{11}$$

and  $\mu_i$  are coordinates on  $\mathbb{R}^3$  subject to the constraint  $\mu_i\mu_i = 1$  that defines the unit 2-sphere. In terms of the left-invariant 1-forms  $R_i$  and  $P_a$  of this paper, we have the correspondences:

$$(D\mu^i)^2 = 4(R_1^2 + R_2^2), \quad \sigma = 2R_3, \quad d\Omega_4^2 = P_a^2, \tag{12}$$

together with  $dA^i + (1/2)\epsilon_{ijk}A^j \wedge A^k = J^i$ .

We can derive first-order differential equations for the three metric functions  $a, b$  and  $c$ , which will imply Spin(7) holonomy, by requiring the existence of a closed self-dual 4-form that has the symmetries of the octonionic structure constants. It is straightforward to see that an appropriate 4-form, invariant under the isometries of the metric and with the required octonionic structure, is given by

$$\Phi_{(4)} = -e^0 \wedge e^1 \wedge e^2 \wedge e^3 - e^8 \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \frac{1}{2}\epsilon_{ijk}e^{\hat{i}} \wedge e^{\hat{j}} \wedge \hat{J}^k + e^8 \wedge e^{\hat{i}} \wedge \hat{J}^i \tag{13}$$

with  $\hat{J}^i \equiv c^2 J^i = (1/2)J_{ab}^i e^a \wedge e^b, 1 \leq i \leq 3$ , where

$$J^i \equiv P_0 \wedge P_i + \frac{1}{2}\epsilon_{ijk}P_j \wedge P_k \tag{14}$$

are the three self-dual quaternionic-Kähler 2-forms on the unit  $S^4$  metric  $P_a^2$ .

From (7) we see that  $d\Phi_{(4)} = 0$  implies the first-order equations:

$$\dot{a} = 1 - \frac{b}{2a} - \frac{a^2}{c^2}, \quad \dot{b} = \frac{b^2}{2a^2} - \frac{b^2}{c^2}, \quad \dot{c} = \frac{a}{c} + \frac{b}{2c}. \tag{15}$$

These equations imply that the metric (8) has Spin(7) holonomy, and also, therefore, that it is Ricci-flat.

It is sometimes convenient to express the equations for  $a, b$  and  $c$  as a Lagrangian system. We find that the equations for Ricci-flatness of (8) can be derived by requiring that  $L \equiv T - V$  be stationary with respect to variations of  $\alpha, \beta$  and  $\gamma$ , where

$$T = 2\alpha'^2 + 12\gamma'^2 + 4\alpha'\beta' + 8\beta'\gamma' + 16\alpha'\gamma', \\ V = \frac{1}{2}b^2c^4(4\alpha^6 + 2a^4b^2 - 24a^4c^2 - 4a^2c^4 + b^2c^4), \tag{16}$$

together with the constraint  $T + V = 0$ . Here a prime denotes a derivative with respect to a new radial variable  $\rho$ , defined by  $dt = a^2bc^4 d\rho$ , and we have also defined  $\alpha = \log a, \beta = \log b, \gamma = \log c$ .

We find that the potential  $V$  can be derived from a superpotential, which we denote by  $W$ . Writing  $T = (1/2)g_{ij}(d\alpha^i/d\rho)(d\alpha^j/d\rho)$ , where  $\alpha_i = (\alpha, \beta, \gamma)$ , we have  $V = -(1/2)g_{ij}(\partial W/\partial\alpha^i)(\partial W/\partial\alpha^j)$ , where

$$W = bc^2(4a^3 + 2a^2b + 4ac^2 - bc^2). \tag{17}$$

From this, we can obtain the first-order equations  $d\alpha^i/d\eta = g^{ij}\partial W/\partial\alpha^j$ . Expressed back in terms of the original radial variable  $t$  introduced in (8), these equations are precisely those given in (15).

To summarise, we have the following proposition.

**Proposition 2.1.** *Metrics of the form (8) are Ricci-flat if and only if  $a, b$  and  $c$  satisfy the Euler–Lagrange equations following from (16). The gradient flow defined by (15) provides three first integrals which solve these second-order Euler–Lagrange equations.*

Before proceeding to find new solutions to these first-order equations, we can first verify that the previous Spin(7) metric (1) is indeed a solution. Also, we may observe that one of the seven-dimensional metrics of  $G_2$  holonomy has principal orbits that are  $\mathbb{C}\mathbb{P}^3$ , viewed as an  $S^2$  bundle over  $S^4$ , and is given, in local coordinates, by [1,5]:

$$ds_7^2 = \left(1 - \frac{\ell^4}{r^4}\right)^{-1} dr^2 + r^2 \left(1 - \frac{\ell^4}{r^4}\right) (R_1^2 + R_2^2) + \frac{1}{2} r^2 P_a^2. \tag{18}$$

This therefore gives a solution in  $D = 8$  of the form  $d\hat{s}_8^2 = ds_7^2 + d\varphi^2$ , and it can be described within the framework of our first-order equation (15) by first rescaling  $b \rightarrow \lambda b$ , and then sending  $\lambda$  to zero, so that  $b = \text{constant}$  is allowed as a solution.

One can also see the specialisations to the previous results described above at the level of the first-order equations themselves. Setting  $a = b$  gives a consistent truncation of (15), yielding  $\dot{a} = (1/2)a^2c^{-2}$ ,  $\dot{c} = (3/2)ac^{-1}$ , which are indeed the first-order equations for the original Spin(7) metrics. On the other hand, sending  $b \rightarrow 0$  in (15) yields a consistent truncation to  $\dot{a} = 1 - a^2c^{-2}$ ,  $\dot{c} = ac^{-1}$ , which are the first-order equations for the metrics of  $G_2$  holonomy whose principal orbits are  $S^2$  bundles over  $S^4$ . (The first-order equations for these two cases can be found, for example, in [3].) Also, we may note that a special solution arises if we set  $b = -a$ , which then implies  $a = -b = \pm c = (1/2)t$ . This is flat space.

Another specialisation of the metric ansatz (8) that makes contact with previous results is to set  $a = c$ , in which case the  $S^2$  bundle over  $S^4$  becomes precisely the usual  $\mathbb{C}\mathbb{P}^3$  Einstein manifold, with its SU(4)-invariant metric. This is incompatible with the first-order equations (15), but it is easily verified that it is consistent with the second-order Einstein equations following from (16). Solutions to these second-order equations then include the eight-dimensional Taub-NUT and Taub-Bolt metrics. The incompatibility with the first-order equations is understandable, since the Taub-NUT and Taub-Bolt 8-metrics do not have special holonomy. Another previously seen solution of the second-order equations with  $a = c$  is the Ricci-flat Kähler metric on the complex line-bundle over  $\mathbb{C}\mathbb{P}^3$ . Although this can arise from a first-order system, it is an inequivalent one that is not related to a specialisation of (15). Its superpotential is  $W = 2a^6 + 6a^4b^2$  [2], with  $T, V$  and  $g_{ij}$  following from setting  $a = c$  in (16). (Other examples of this kind of phenomenon were exhibited recently in [3].)

### 3. General solution of the gradient flow

#### 3.1. The general solution: local analysis

In order to obtain new solutions of the first-order equations (15), we first introduce a new radial coordinate  $r$ , defined in terms of  $t$  by  $dr = b dt$ . After also defining  $f \equiv c^2$ , we find by taking further derivatives of the first-order equations (15) that  $f$  must satisfy the third-order equation:

$$2f^2 f''' + 2f(f' - 3)f'' - (f' + 1)(f' - 1)(f' - 3) = 0 \tag{19}$$

which can be expressed in the “factorised” form:

$$fQ' - (f' + 1)Q = 0, \quad (20)$$

where  $Q \equiv 2fW' + (f' - 3)W$  and  $W \equiv f' - 1$ . The remaining metric functions are then given by solving

$$a' = \frac{f' - 2}{2a} - \frac{(f' - 1)a}{2f}, \quad b = \frac{2a}{f' - 1}. \quad (21)$$

Naively there now appear to be four constants of integration in total rather than the expected three, but the extra one is eliminated by substituting the solutions back into (15). In fact for a generic solution, where  $Q$  itself is non-zero, the solution for  $a$ , and hence for  $b$ , can be written entirely algebraically in terms of  $f$ , with

$$a^2 = \frac{(f' - 1)(f' - 3)f}{Q}, \quad b = \frac{2a}{f' - 1}. \quad (22)$$

Thus for a solution where  $Q \neq 0$  the three integration constants for the first-order system (15) are simply the three integration constants for the third-order equation (19), and no further substitution back into (19) is necessary. As we shall see below,  $Q$  is non-vanishing for all but one degenerate solution of (19). Note that two of the three constants of integration are “trivial,” corresponding to a constant shift and rescaling of the radial coordinate.

The general solution to Eq. (19) may be obtained as follows. First, introduce a new radial variable  $\rho$ , and a function  $\gamma(\rho)$ , defined by

$$f(r) = \exp \left[ - \int^\rho \frac{\tilde{\rho} d\tilde{\rho}}{\gamma(\tilde{\rho})} \right], \quad \frac{df}{dr} = \rho, \quad (23)$$

implying also that

$$\frac{d^2 f}{dr^2} = -\frac{\gamma}{f}, \quad \frac{d^3 f}{dr^3} = \frac{1}{f^2} \left( \frac{\rho}{\gamma} + \gamma \frac{d\gamma}{d\rho} \right). \quad (24)$$

(We assume here, and in the rest of this subsection, that  $d f/d r$  is not a constant, and so  $\rho$  is a good radial variable. The special cases where  $d f/d r$  is a constant are included in the discussion in Section 3.2.) Eq. (19) now reduces to

$$2\gamma \frac{d\gamma}{d\rho} + 6\gamma = (1 - \rho^2)(3 - \rho). \quad (25)$$

The further replacement of  $\gamma$  by  $z$ , and  $\rho$  by  $v$ , defined by

$$z \equiv \frac{(1 - \rho)^2}{2(1 - \rho - \gamma)}, \quad v \equiv \rho - 1 \quad (26)$$

turns (25) into

$$2z(1 - z^2) \frac{dv}{dz} = v + 2z. \quad (27)$$

The solution to this equation can be written in terms of the hypergeometric function as

$$v = \frac{2k\sqrt{z}}{(1-z^2)^{1/4}} - 2z {}_2F_1 \left[ 1, \frac{1}{2}; \frac{5}{4}; 1-z^2 \right]. \tag{28}$$

Retracing the steps of the various redefinitions, we see that by using  $z$  as the radial variable the general solution (with  $df/dr$  not a constant) for the Ricci-flat metric can be written as

$$a^2 = \frac{(v-2)zf}{(1+z)v}, \quad b = \frac{2a}{v}, \quad c^2 \equiv f \left( \frac{1+z}{1-z} \right)^{1/2} \exp \left[ \int^z \frac{d\tilde{z}}{v(\tilde{z})(1-\tilde{z}^2)} \right]. \tag{29}$$

The coordinate  $r$  is given in terms of  $z$  by

$$dr = \frac{f dz}{v(z)(1-z^2)} \tag{30}$$

and so we have the following proposition.

**Proposition 3.1.** *The local Ricci-flat metrics arising from the gradient flow (15) are given by*

$$ds_8^2 = \frac{vfdz^2}{4z(1-z^2)(1-z)(v-2)} + \frac{4(v-2)zf}{(1+z)v} (R_1^2 + R_2^2) + \frac{16(v-2)zf}{(1+z)v^3} R_3^2 + fP_a^2 \tag{31}$$

with  $v$  defined by (28) and  $f$  defined by (29).

Note also that  $k$  in the solution (28) for  $v(z)$  is the non-trivial third constant of integration of the original first-order system (15). In obtaining this general solution we have assumed that  $Q$  is non-zero, so that  $a$  can be obtained using (22). In fact if  $Q$  is zero it can easily be seen that unless in addition  $df/dr$  is a constant (which we have excluded from the analysis in this subsection), then after using (21) and substituting back into (15), the metric functions  $a$ ,  $b$  and  $c$  would all vanish. Thus the only additional solutions to (15), other than those described by (31), are those with  $df/dr = \text{constant}$ , and these are included in the discussion in Section 3.2.

### 3.2. Special globally defined solutions

As we shall show latter, the general Ricci-flat metrics obtained in Section 3.1 include one-parameter families of examples that are complete on manifolds with the topology of the bundle of chiral spinors over  $S^4$ . The parameter in question is a non-trivial one, as opposed to the two trivial parameters associated with a constant shift and scaling of the radial coordinate. Before discussing these families of complete metrics, we shall first discuss some simple solutions of the first-order equations (15). It is easier to discuss these in terms of the original radial variable  $r$  used in writing (19). After absorbing a trivial constant shift of the radial variable we can write down three elementary solutions of (19), namely

$$f = -r, \quad f = 3r, \quad f = r + \frac{r^2}{2\ell^2}, \tag{32}$$

where  $\ell$  is a constant. The first two solutions here are of the type where  $df/dr$  is a constant, which were excluded in the general analysis Section 3.1:

- The solution with  $f = -r$  has  $Q = 8$  and so we can use (22), to find

$$a^2 = -r, \quad b = -a, \quad c^2 = r. \tag{33}$$

It follows from (8) that the metric in this case is just the trivial flat metric on  $\mathbb{R}^8$ , with  $r \leq 0$ . In terms of the description using  $z$  and  $v$  introduced in Section 3.1, it corresponds to a degenerate solution at the point  $(z, v) = (1, -2)$ .

- The solution with  $f = 3r$  has  $Q = 0$ , and so here we must solve for  $b$  using (21). After making a coordinate transformation  $r \rightarrow 3r^2/20$ , this solution is

$$a^2 = \frac{3}{10}r \left[ 1 - \left(\frac{r_0}{r}\right)^{10/3} \right], \quad b = a, \quad c^2 = \frac{9}{20}r^2. \tag{34}$$

This can be recognised as the previously known complete metric of Spin(7) holonomy [1,5], as given in (1). (The trivial scaling constant  $r_0$  arose here in the integration of the equation for  $b$  in (21).) Note that because this solution has  $f' = \text{constant}$ , it is not contained within the general analysis of Section 3.1, except as a singular limit that corresponds to the point  $(z, v) = (-1, +2)$ .

- The third elementary solution in (32),  $f = r + r^2/(2\ell^2)$ , gives rise to our first examples of new complete metrics of Spin(7) holonomy. After a coordinate transformation  $r \rightarrow -\ell(r + \ell)$ , the metric in local coordinates becomes

$$ds_8^2 = \frac{(r + \ell)^2 dr^2}{(r + 3\ell)(r - \ell)} + \frac{4\ell^2(r + 3\ell)(r - \ell)}{(r + \ell)^2} R_3^2 + (r + 3\ell)(r - \ell)(R_1^2 + R_2^2) + \frac{1}{2}(r^2 - \ell^2)P_a^2. \tag{35}$$

Assuming that the constant  $\ell$  is positive, it is evident that  $r$  should lie in the range  $r \geq \ell$ . We can analyse the behaviour near  $r = \ell$  by defining a new radial coordinate  $\rho$ , where  $\rho^2 = 4\ell(r - \ell)$ . Near  $\rho = 0$  the metric approaches

$$ds_8^2 \approx d\rho^2 + \rho^2(R_1^2 + R_2^2 + R_3^2 + \frac{1}{4}P_a^2). \tag{36}$$

The quantity  $R_1^2 + R_2^2 + R_3^2 + (1/4)d\Omega_4^2$  is precisely the metric on the unit 7-sphere, and so we see that near  $r = \ell$  the metric  $ds_8^2$  smoothly approaches flat  $\mathbb{R}^8$ . At large  $r$  the function  $b$ , which is the radius in the  $U(1)$  direction  $R_3$ , approaches a constant, and so the metric approaches an  $S^1$  bundle over a 7-metric. This 7-metric is of the form of a cone over  $\mathbb{C}\mathbb{P}^3$  (described as the  $S^2$  bundle over  $S^4$ ) in this asymptotic region. The manifold of this new Spin(7) metric, which we are denoting by  $\mathbb{A}_8$ , is topologically  $\mathbb{R}^8$ . In terms of the description in Section 3.1, this solution corresponds to a trajectory in the  $(z, v)$  plane with  $z = 1$ , and  $v$  running from  $v = -2$  (at the origin) to  $v = -\infty$  (in the asymptotic region). Thus we have the following proposition.

**Proposition 3.2.** *The metric  $\mathbb{A}_8$  given by (35) with  $r > \ell > 0$  admits a smooth complete non-singular extension to  $\mathbb{R}^8$ .*

We shall use the acronym AC to denote asymptotically conical manifolds. Thus asymptotically our new metrics behave like a circle bundle over an AC manifold in which the



length of the  $U(1)$  fibres tends to a constant. The acronym ALF is already in use to describe metrics which tend to a  $U(1)$  bundle over an asymptotically Euclidean or asymptotically locally Euclidean metric with the length of the fibres tending to a constant. We shall therefore adopt the acronym ALC to denote manifolds where the base space of the circle bundle is asymptotically conical.

Ricci-flat ALC metrics, although not with special holonomy, have already been encountered. For example, the higher-dimensional Taub-NUT metric is defined on  $\mathbb{R}^{2n}$  for all  $n$  and it is ALC with the base of the cone being  $\mathbb{C}\mathbb{P}^{n-1}$ . A closely related example is the Taub-Bolt metric which has the same asymptotics but is defined on a line-bundle over  $\mathbb{C}\mathbb{P}^{n-1}$ . However, the metric on the base of the cone in this case (with  $n = 4$ ) is the Fubini–Study metric on  $\mathbb{C}\mathbb{P}^3$ , which is quite different from that of the “squashed” Einstein metric on  $\mathbb{C}\mathbb{P}^3$  in our new metrics. A discussion of ALE Spin(7) manifolds based on the idea of blowing up orbifolds has been given in [6]. As far as we are aware, no explicit examples of this kind have yet been found.

We get a different complete manifold, which we are denoting by  $\mathbb{B}_8$ , if we take  $r$  to be negative. It is easier to discuss this by instead setting  $\ell = -\tilde{\ell}$ , where  $\tilde{\ell}$  and  $r$  are taken to be positive. Thus instead of (35) we now have

$$\begin{aligned}
 ds_8^2 = & \frac{(r - \tilde{\ell})^2 dr^2}{(r - 3\tilde{\ell})(r + \tilde{\ell})} + \frac{4\tilde{\ell}^2(r - 3\tilde{\ell})(r + \tilde{\ell})}{(r - \tilde{\ell})^2} R_3^2 + (r - 3\tilde{\ell})(r + \tilde{\ell})(R_1^2 + R_2^2) \\
 & + \frac{1}{2}(r^2 - \tilde{\ell}^2)P_a^2.
 \end{aligned}
 \tag{37}$$

This time, we have  $r \geq 3\tilde{\ell}$ . Defining  $\rho^2 = 4\tilde{\ell}(r - 3\tilde{\ell})$ , we find that near  $r = 3\tilde{\ell}$  the metric has the form:

$$ds_8^2 \approx d\rho^2 + \rho^2(R_1^2 + R_2^2 + R_3^2) + \tilde{\ell}^2 P_a^2.
 \tag{38}$$

The quantity  $(R_1^2 + R_2^2 + R_3^2)$  is the metric on the unit 3-sphere, and so in this case we find that the metric smoothly approaches  $\mathbb{R}^4 \times S^4$  locally, at small distance. The large-distance behaviour is the same as for the previous case (35). In the  $(z, v)$  plane of Section 3.1, this solution corresponds to a trajectory with  $z = 1$ , and  $v$  running from  $v = +2$  (at the origin) to  $v = +\infty$  (in the asymptotic region). Thus we have the following proposition.

**Proposition 3.3.** *The metric  $\mathbb{B}_8$  given by (37) with  $r > 3\tilde{\ell} > 0$  admits a smooth complete non-singular extension to the chiral spin bundle over  $S^4$ .*

Again we have a complete non-compact ALC metric with Spin(7) holonomy with the same base. At short distance, it has the same structure as the previously known metric of Spin(7) holonomy, obtained in [5].

We can think of the new manifold  $\mathbb{A}_8$  as providing a smooth interpolation between Euclidean 8-space at short distance, and an  $S^1$  bundle over  $\mathcal{M}_7$  at large distance, while  $\mathbb{B}_8$  provides an interpolation between the previous Spin(7) manifold of [1,5] at short distance and the  $S^1$  bundle over  $\mathcal{M}_7$  at large distance. Here  $\mathcal{M}_7$  denotes the 7-manifold of  $G_2$  holonomy that is the  $R^3$  bundle over  $S^4$  [1,5].

### 3.3. General globally defined solutions

Having discussed some special solutions of (15) in Section 3.2, and having seen that they include new complete metrics of Spin(7) holonomy, we now turn to a discussion of the global structure of the general solutions (31).

In order to recognise the solutions that give rise to complete non-singular metrics, it is helpful first to study the phase-plane diagram for the first-order equation (27), which can be expressed as

$$\frac{dz}{d\tau} = 2z(1 - z^2), \quad \frac{dv}{d\tau} = v + 2z, \tag{39}$$

where  $\tau$  is an auxiliary “time” parameter. The solutions can be studied by looking at the flows generated by the 2-vector field  $\{dz/d\tau, dv/d\tau\} = \{2z(1 - z^2), v + 2z\}$  in the  $(z, v)$  plane. For any such flow, it is then necessary to investigate the global structure of the associated metric (31) for regularity.

We find that regular solutions can arise in the following four cases, namely

- (1)  $\mathbb{A}_8$  :  $z = 1$  (fixed);  $v = -2$  to  $-\infty$ ,
- (2)  $\mathbb{B}_8$  :  $z = 1$  (fixed);  $v = +2$  to  $+\infty$ ,
- (3)  $\mathbb{B}_8^-$  :  $z_0 \leq z \leq 1$ ;  $v = +2$  to  $+\infty$ ,  $0 < z_0 < 1$ ,
- (4)  $\mathbb{B}_8^+$  :  $1 \leq z \leq z_0$ ;  $v = +2$  to  $+\infty$ ,  $1 < z_0 < \infty$  or  $z_0 < -1$ ;  
(see discussion below). (40)

Note that  $v = \pm\infty$  corresponds to the asymptotic large-distance region, and in all four cases the metrics have similar asymptotic structures, precisely as we have already seen in the  $\mathbb{A}_8$  and  $\mathbb{B}_8$  cases. The point  $v = -2, z = 1$  corresponds to the short-distance behaviour of the  $\mathbb{A}_8$  metric, approaching Euclidean  $\mathbb{R}^8$  at the origin where the  $S^7$  principal orbits degenerate to a point. When  $v = 2$ , on the other hand, we have the short-distance behaviour seen in the  $\mathbb{B}_8$  metric, approaching  $\mathbb{R}^4 \times S^4$  locally. In fact solution (1) is the metric (35) on  $\mathbb{A}_8$  found in Section 3.2, and solution (2) is the metric (37) on  $\mathbb{B}_8$  found there also. These both have  $k = 0$  in (28).

Solution (3) arises when  $k$  is any positive number, with  $z_0$  being the corresponding value of  $z$  at which  $v(z_0) = 2$ , with  $0 < z_0 < 1$ . The value of  $z_0$  is correlated with the value of  $k$ , ranging from  $z_0 = 0$  for  $k = \infty$ , to  $z_0 = 1$  for  $k = 0$ .

Near  $z = 1$  it follows from (28) that we shall have

$$v = 2^{3/4}k(1 - z)^{-1/4} - 2 + \dots, \quad f = c_0(1 - z)^{-1/2} + \dots, \tag{41}$$

where  $c_0$  is an arbitrary constant of integration. Defining  $y \equiv (2c_0)^{-1/2}(1 - z)^{-1/4}$ , we see that as  $z \rightarrow 1$  we shall have  $y \rightarrow \infty$  and

$$ds_8^2 \approx dy^2 + y^2(R_1^2 + R_2^2) + \frac{1}{2}y^2 P_a^2 + \frac{2\sqrt{2}c_0}{k^2}R_3^2 \tag{42}$$

and so this more general metric has the same large-distance asymptotic form as do  $\mathbb{A}_8$  and  $\mathbb{B}_8$ . Near  $z = z_0$  we shall have  $v(z) = 2 + v'(z_0)(z - z_0) + \dots$ , and defining a new radial

coordinate  $x$  by  $(z - z_0) = (1/4)x^2$  near  $z = z_0$ , we shall have

$$ds_8^2 \approx \frac{f_0}{2z_0(1-z_0^2)(1-z_0)v'(z_0)} [dx^2 + v'(z_0)^2 z_0^2 (1-z_0)^2 x^2 (R_1^2 + R_2^2 + R_3^2)] + f_0 P_a^2, \tag{43}$$

where  $f_0$  is the value of  $f$  at  $z = z_0$ . From (27) we have that  $z_0(1 - z_0)v'(z_0) = 1$ , and so we see from (43) that at short distance the metric (43) approaches  $\mathbb{R}^4 \times S^4$  locally. Thus these more general solution (3) in (40) with  $k > 0$  is complete on a manifold that is very similar to the manifold  $\mathbb{B}_8$  of the solution (37), with an  $S^4$  bolt at  $z = z_0$ . Here, the trajectory in the  $(z, v)$  plane runs from  $(z_0, 2)$  to  $(1, \infty)$ . We shall denote the solution by  $\mathbb{B}_8^-$ , where the superscript indicates that  $z$  starts from a value  $z_0 < 1$  at short distance, flowing to  $z = 1$  asymptotically. For the case  $k = 0$ , which leads to the metrics (35) and (37), the quantity  $z$  is not a good choice for the radial coordinate, since it is fixed at  $z = 1$ . This case can be regarded as a singular limit within the general formalism we are using here. Specifically, if we let  $z = 1 - 16\epsilon^4 \tilde{\ell}^4 (r + \tilde{\ell})^{-4}$ ,  $k = 2^{1/4}\epsilon$ , and choose the integration constant in (29) so that  $f = (1/2)(r^2 - \tilde{\ell}^2)$ , then upon sending  $\epsilon$  to zero we recover the metric (37).

Solution (4) arises in the case where at large distance  $z$  now approaches 1 from above, and again the flow runs from an  $S^4$  bolt at which  $v(z_0) = 2$ , to the asymptotic region as  $z$  approaches 1. There are two possibilities, with  $z_0$  either being greater than 1, or else  $z_0$  is less than  $-1$ . In the latter case  $z$  then runs from  $z_0$  at the bolt, through  $z = -\infty$  to  $+\infty$ , and then down to  $z = 1$  in the asymptotic region. It is useful now to make another change of radial coordinate, and define  $y \equiv 1/z$ . This allows the two regions for  $z_0$  to be combined. The solution for  $v$  may now be written as

$$v = (1 - y^2)^{-1/4} (\kappa + y {}_2F_1[\frac{1}{2}, \frac{3}{4}; \frac{3}{2}; y^2]). \tag{44}$$

The  $y$  coordinate then ranges from  $y = y_0$  at the bolt to  $y = 1$  at infinity, and  $-1 \leq y_0 \leq 1$ . The integration constant  $\kappa$  is determined in terms of  $y_0$  by the requirement that  $v = 2$  at  $y = y_0$ . It can range between  $\kappa = 2\sqrt{\pi}\Gamma(5/4)/\Gamma(3/4)$ , corresponding to  $y_0 = -1$ , and  $\kappa = -2\sqrt{\pi}\Gamma(5/4)/\Gamma(3/4)$ , corresponding (by taking a suitable limit) to  $y_0 = +1$ .

A similar analysis to that for solution (3) above now shows that the metric in solution (4) smoothly approaches  $\mathbb{R}^4 \times S^4$  locally at  $y = y_0$ , and that it has the same asymptotic behaviour as the previous examples. We shall denote this solution by  $\mathbb{B}_8^+$ . Note that the simple solution  $\mathbb{B}_8^+$  in (37) can be viewed as the  $k \rightarrow 0$  or  $\kappa \rightarrow -2\sqrt{\pi}\Gamma(5/4)/\Gamma(3/4)$  limit of the more complicated  $\mathbb{B}_8^-$  or  $\mathbb{B}_8^+$  solutions, respectively.

We observed at the end of Section 2 that a particular example of a solution of the first-order equations (15) is the direct product metric  $ds_8^2 = ds_7^2 + d\Phi^2$ , where  $ds_7^2$  is the Ricci-flat 7-metric of  $G_2$  holonomy on the  $\mathbb{R}^3$  bundle over  $S^4$  [1,5], and  $\varphi$  is a coordinate on a circle. We are now in a position to see how this solution can arise as a limit of our new Spin(7) metrics. Specifically, it arises as the  $k \rightarrow \infty$  limit of solution (3) listed in (40). This is the limit where the constant  $z_0$ , which sets the lower limit for the range  $z_0 \leq z \leq 1$  for  $z$ , becomes zero. At the same time as sending  $k$  to infinity, we can rescale the fibre coordinate  $\varphi$  appearing in  $R_3 = (1/2)d\varphi + \dots$ , according to  $\varphi \rightarrow k\varphi$ .

From (28) and (29) it follows that when  $k$  becomes very large we shall have

$$V \rightarrow \frac{2k\sqrt{z}}{(1-z^2)^{1/4}}, \quad f \rightarrow \left(\frac{1+z}{1-z}\right)^{1/2} \tag{45}$$

and so in the limit of infinite  $k$  the metric (31) becomes

$$ds_8^2 = \frac{dz^2}{4z(1-z)^2(1-z^2)^{1/2}} + \frac{4z}{(1-z^2)^{1/2}}(R_1^2 + R_2^2) + \left(\frac{1+z}{1-z}\right)^{1/2} P_a^2 + d\varphi^2. \tag{46}$$

Defining a new radial coordinate  $r$  by  $r^4 = (1+z)(1-z)^{-1}$ , we see that this becomes  $ds_8^2 = ds_7^2 + d\varphi^2$ , where

$$ds_7^2 = \frac{2dr^2}{1-r^{-4}} + 2r^2(1-r^{-4})(R_1^2 + R_2^2) + r^2 P_a^2. \tag{47}$$

This can be recognised as the metric of  $G_2$  holonomy on the manifold  $\mathcal{M}_7$  of the  $\mathbb{R}^3$  bundle over  $S^4$ , which was constructed in [1,5]. (If we had not rescaled the fibre coordinate  $\varphi$  by a factor of  $k$  before taking the limit  $k \rightarrow \infty$ , the radius of the  $S^1$  would have tended to zero, this limit being taken in the sense of Gromov–Hausdorff.) Thus the family of new Spin(7) manifolds that we are denoting by  $\mathbb{B}_8^-$  has a non-trivial parameter  $k$  such that the  $k = \infty$  limit degenerates to  $\mathcal{M}_7 \times S^1$ , while the  $k = 0$  limit reduces to the case  $\mathbb{B}_8$  given by (37).

#### 4. Parallel spinor and calibrating 4-form

The fact that the metric (8), together with (15), has Spin(7) holonomy implies, and is implied by, the existence of a globally defined parallel spinor field  $\eta$ , satisfying  $D\eta = 0$ , where  $D \equiv d + (1/4)\hat{\omega}_{AB}\Gamma_{AB}$  is the Lorentz-covariant exterior derivative that acts on spinors in eight dimensions. Here  $\Gamma_{AB} \equiv (1/2)(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A)$ , and  $\Gamma_A$  are the Dirac matrices that generate the Clifford algebra in eight dimensions.

After a straightforward calculation, we find that  $D$  is given by

$$\begin{aligned} D = d + e^0 & \left( \frac{\dot{c}}{c}\Gamma_{08} - \frac{a}{4c^2}\Gamma_{1\hat{1}} - \frac{a}{4c^2}\Gamma_{2\hat{2}} - \frac{b}{4c^2}\Gamma_{3\hat{3}} \right) \\ & + e^1 \left( \frac{\dot{c}}{c}\Gamma_{18} + \frac{a}{4c^2}\Gamma_{0\hat{1}} + \frac{a}{4c^2}\Gamma_{3\hat{2}} - \frac{b}{4c^2}\Gamma_{2\hat{3}} \right) \\ & + e^2 \left( \frac{\dot{c}}{c}\Gamma_{28} + \frac{a}{4c^2}\Gamma_{0\hat{2}} - \frac{a}{4c^2}\Gamma_{3\hat{1}} + \frac{b}{4c^2}\Gamma_{1\hat{3}} \right) \\ & + e^3 \left( \frac{\dot{c}}{c}\Gamma_{38} + \frac{a}{4c^2}\Gamma_{2\hat{1}} + \frac{a}{4c^2}\Gamma_{1\hat{2}} - \frac{b}{4c^2}\Gamma_{0\hat{3}} \right) \\ & + e^{\hat{1}} \left( \frac{\dot{a}}{2a}\Gamma_{\hat{1}8} + \left( \frac{a}{4c^2} - \frac{1}{4a} \right) (\Gamma_{0\hat{1}} + \Gamma_{2\hat{3}}) - \frac{b}{4a^2}\Gamma_{\hat{2}\hat{3}} \right) \end{aligned}$$

$$\begin{aligned}
 &+ e^{\hat{2}} \left( \frac{\dot{a}}{2a} \Gamma_{\hat{2}8} + \left( \frac{a}{4c^2} - \frac{1}{4a} \right) (\Gamma_{02} + \Gamma_{\hat{3}1}) - \frac{b}{4a^2} \Gamma_{\hat{3}\hat{1}} \right) \\
 &+ e^{\hat{3}} \left( \frac{\dot{b}}{2b} \Gamma_{\hat{3}8} + \left( \frac{b}{4c^2} - \frac{1}{4b} \right) (\Gamma_{03} + \Gamma_{12}) + \left( \frac{b}{4a^2} - \frac{1}{2b} \right) \Gamma_{\hat{1}\hat{2}} \right). \tag{48}
 \end{aligned}$$

Substituting in the first-order equations (15), this becomes

$$\begin{aligned}
 D = d &+ e^0 \left( \frac{a}{4c^2} (2\Gamma_{08} - \Gamma_{\hat{1}\hat{1}} - \Gamma_{\hat{2}\hat{2}}) + \frac{b}{4c^2} (\Gamma_{08} - \Gamma_{\hat{3}\hat{3}}) \right) \\
 &+ e^1 \left( \frac{a}{4c^2} (2\Gamma_{18} + \Gamma_{0\hat{1}} + \Gamma_{\hat{3}\hat{2}}) + \frac{b}{4c^2} (\Gamma_{18} - \Gamma_{\hat{2}\hat{3}}) \right) \\
 &+ e^2 \left( \frac{a}{4c^2} (2\Gamma_{28} + \Gamma_{0\hat{2}} + \Gamma_{\hat{3}\hat{1}}) + \frac{b}{4c^2} (\Gamma_{28} - \Gamma_{\hat{1}\hat{3}}) \right) \\
 &+ e^3 \left( \frac{a}{4c^2} (2\Gamma_{38} - \Gamma_{\hat{1}\hat{2}} + \Gamma_{\hat{2}\hat{1}}) + \frac{b}{4c^2} (\Gamma_{38} - \Gamma_{0\hat{3}}) \right) \\
 &+ e^{\hat{1}} \left( \left( \frac{1}{4a} - \frac{a}{4c^2} \right) (2\Gamma_{\hat{1}8} - \Gamma_{01} - \Gamma_{23}) - \frac{b}{4a^2} (\Gamma_{\hat{1}8} + \Gamma_{\hat{2}\hat{3}}) \right) \\
 &+ e^{\hat{2}} \left( \left( \frac{1}{4a} - \frac{a}{4c^2} \right) (2\Gamma_{\hat{2}8} - \Gamma_{02} - \Gamma_{31}) - \frac{b}{4a^2} (\Gamma_{\hat{2}8} + \Gamma_{\hat{3}\hat{1}}) \right) \\
 &+ e^{\hat{3}} \left( \frac{b}{4a^2} (\Gamma_{\hat{3}8} + \Gamma_{\hat{1}\hat{2}}) - \frac{b}{4c^2} (2\Gamma_{\hat{3}8} - \Gamma_{03} - \Gamma_{12}) - \frac{1}{4b} (2\Gamma_{\hat{1}\hat{2}} + \Gamma_{03} + \Gamma_{12}) \right). \tag{49}
 \end{aligned}$$

It is now straightforward to see that in this frame, a spinor  $\eta$  satisfies  $D\eta = 0$  if it has constant components, and if in addition it satisfies the projection conditions:

$$(2\Gamma_{08} - \Gamma_{\hat{1}\hat{1}} - \Gamma_{\hat{2}\hat{2}})\eta = 0, \quad (\Gamma_{08} - \Gamma_{\hat{3}\hat{3}})\eta = 0, \quad (2\Gamma_{18} + \Gamma_{0\hat{1}} + \Gamma_{\hat{3}\hat{2}})\eta = 0. \tag{50}$$

These conditions define a unique spinor, up to overall scale, thus providing another proof that the metrics have Spin(7) holonomy.

The covariantly constant self-dual 4-form  $\Phi$  given in (13), known as the Cayley form, provides a calibration of the Spin(7) manifold. Thus we have

$$|\Phi(X_1, X_2, X_3, X_4)| \leq 1, \tag{51}$$

where  $(X_1, X_2, X_3, X_4)$  denotes any quadruple of orthonormal vectors. This can be seen from (13), or else by noting that the components of  $\Phi$  can be expressed in terms of the parallel spinor  $\eta$  as  $\Phi_{ABCD} = \tilde{\eta} \Gamma_{ABCD} \eta$ . A calibrated submanifold, or Cayley submanifold,  $\Sigma$ , is one where for each point of  $\Sigma$ :

$$|\Phi(X_1, X_2, X_3, X_4)| = 1, \tag{52}$$

where the orthonormal vectors  $X_i$  are everywhere tangent to  $\Sigma$ . By inspecting (13) we therefore see that the  $S^4$  zero section of the bundle of chiral spinors is a Cayley submanifold, and hence it is volume minimising in its homology class.

Thus in summary we have the following proposition.

**Proposition 4.1.** *The gradient flow equations (15) are the necessary and sufficient conditions for the local metric (8) to have holonomy Spin(7). The covariantly constant spinor is defined by (50), and the Cayley 4-form is given by (13).*

### 5. $L^2$ -normalisable harmonic 4-forms in $\mathbb{A}_8$ and $\mathbb{B}_8$

In this section, derive the equations for harmonic 4-forms in the Spin(7) manifolds. We obtain explicit  $L^2$ -normalisable harmonic 4-forms for each of the new Spin(7) 8-manifolds  $\mathbb{A}_8$  and  $\mathbb{B}_8$ . Specifically, we obtain one such 4-form, which is anti-self-dual, for the manifold  $\mathbb{A}_8$  that is topologically  $\mathbb{R}^8$ , and two such 4-forms, one of each duality, for the manifold  $\mathbb{B}_8$  of the chiral spin bundle over  $S^4$ .

The structure of the harmonic 4-forms turns out to be closely related to that of the calibrating 4-form  $\Phi$  given in (13). Thus we define

$$G_{(4)}^\pm = \omega_{(4)} \pm *\omega_{(4)}, \tag{53}$$

where

$$\omega_{(4)} \equiv u_1 e^0 \wedge e^1 \wedge e^2 \wedge e^3 - u_2 (e^{\hat{2}} \wedge e^{\hat{3}} \wedge \hat{J}^1 + e^{\hat{3}} \wedge e^{\hat{1}} \wedge \hat{J}^2) + u_3 e^{\hat{1}} \wedge e^{\hat{2}} \wedge \hat{J}^3. \tag{54}$$

$G_{(4)}^\pm$  will be harmonic if  $dG_{(4)}^\pm = 0$ . This implies that

$$\begin{aligned} \pm \frac{d(c^4 u_1)}{dt} - 2bc^2 u_2 + 4ac^2 u_3 &= 0, & \pm \frac{d(a^2 c^2 u_2)}{dt} - a^2 b u_1 + bc^2 u_2 + 2ac^2 u_3 &= 0, \\ \pm \frac{d(abc^2 u_3)}{dt} + a^2 b u_1 + bc^2 u_2 &= 0, \end{aligned} \tag{55}$$

where the  $\pm$  signs correspond to self-dual and anti-self-dual, respectively. (The Cayley form given in (13) is a particular solution, corresponding to taking  $u_1 = u_2 = -1, u_3 = 1$ .)

In the case of the new Spin(7) manifolds  $\mathbb{A}_8$  and  $\mathbb{B}_8$ , with their simple metrics (35) and (37), we can now obtain explicit results for  $L^2$  harmonic 4-forms. In the remainder of this section, we shall for convenience set the scale parameters  $\ell$  and  $\tilde{\ell}$  in the metrics (35) and (37) to unity. Care must be exercised when taking the square roots of  $a^2, b^2$  and  $c^2$  in the metrics (35) and (37), if one wants the functions  $a, b$  and  $c$  to solve precisely the first-order equations (15), since these equations are sensitive to the signs of  $a, b$  and  $c$ . (Of course there are equivalent first-order equations that differ by precisely these sign factors, and which also imply solutions of the Einstein equations.) We are assuming here that the signs are chosen so that precisely (15) are satisfied. This can be achieved by taking all square roots to be positive, except for  $b$  in the case of (35) on  $\mathbb{A}_8$ .

For the metric (35) on the manifold  $\mathbb{A}_8$  that is topologically  $\mathbb{R}^8$ , we find that there is a normalisable harmonic 4-form that is anti-self-dual, i.e., the lower choice of the sign is used

in (53) and (55). The solution is given by

$$u_1 = \frac{2}{(r + 1)^3(r + 3)}, \quad u_2 = -\frac{r^2 + 10r + 13}{(r + 1)^3(r + 3)^3}, \quad u_3 = -\frac{2}{(r + 1)^2(r + 3)^3}. \tag{56}$$

The norm of the harmonic anti-self-dual 4-form is then given by

$$|G_{(4)}|^2 = 18(u_1^2 + 2u_2^2 + 4u_3^2) = \frac{96(3r^4 + 44r^3 + 242r^2 + 492r + 339)}{(r + 1)^2(r + 3)^6}. \tag{57}$$

Clearly  $G_{(4)}$  is  $L^2$ -normalisable, and in fact we have  $\int_1^\infty \sqrt{g}|G_{(4)}|^2 dr = 9/4$ . We have chosen the integration constants from (55) appropriately in order to select the solution in  $L^2$ . (There also exists a solution for a self-dual harmonic 4-form. It can be made square integrable at small distance, but there is no choice of integration constants for which it is  $L^2$ -normalisable, owing to its large-distance behaviour.)

For the metric (37) on  $\mathbb{B}_8$ , the bundle of chiral spinors over  $S^4$ , we find that there exists a normalisable harmonic 4-form that is anti-self-dual, i.e., the lower choice of sign is used in (53) and (55). The solution is given by

$$u_1 = \frac{2(r^4 + 8r^3 + 34r^2 - 48r + 21)}{(r - 1)^3(r + 1)^5}, \quad u_2 = \frac{r^4 + 4r^3 - 18r^2 + 52r - 23}{(r - 1)^3(r + 1)^5},$$

$$u_3 = \frac{2(r^2 + 14r - 11)}{(r - 1)^2(r + 1)^5}. \tag{58}$$

The square of the anti-self-dual 4-form is given by

$$|G_{(4)}|^2 = \frac{96(3r^8 + 40r^7 + 252r^6 + 1064r^5 + 2506r^4 - 12936r^3 + 18284r^2 - 10824r + 2379)}{(r - 1)^6(r + 1)^{10}} \tag{59}$$

and its  $L^2$ -normalisability can be seen by noting that  $\int_3^\infty \sqrt{g}|G_{(4)}|^2 dr = 189/16$ .

We also find a second  $L^2$ -normalisable harmonic 4-form in the new Spin(7) manifold  $\mathbb{B}_8$ . This 4-form is self-dual, with the upper sign chosen in (53) and (55) and is given by

$$u_1 = -\frac{2(5r^3 - 9r^2 + 15r - 3)}{(r - 1)^3(r + 1)^4}, \quad u_2 = \frac{(r - 3)(5r^2 - 2r + 1)}{(r - 1)^3(r + 1)^4},$$

$$u_3 = -\frac{2(r - 3)}{(r - 1)^2(r + 1)^4}. \tag{60}$$

In contrast to the previous harmonic 4-forms, there is no linear relation between the functions  $u_1, u_2$  and  $u_3$  here. The magnitude of  $G_{(4)}$  is given by

$$|G_{(4)}|^2 = \frac{96(75r^6 - 350r^5 + 829r^4 - 932r^3 + 885r^2 - 414r + 99)}{(r - 1)^6(r + 1)^8}. \tag{61}$$

It integrates to give  $\int_3^X \sqrt{g}|G_{(4)}|^2 dr = 189/4$ .

It is interesting to note that for the anti-self-dual harmonic 4-form on  $\mathbb{A}_8$ , given by (56), we can write it in terms of a globally defined potential,  $G_{(4)} = dB_{(3)}$ . Specifically, we find that  $B_{(3)}$  can be written as

$$B_{(3)} = -(r-1)^2 \left[ \frac{1}{(r+1)^2} R_1 \wedge R_2 \wedge R_3 + \frac{1}{8(r+3)^2} (R_1 \wedge J^1 + R_2 \wedge J^2) + \frac{(r+5)}{4(r+1)(r+3)^2} R_3 \wedge J^3 \right]. \quad (62)$$

One can see from (35) that this has a vanishing magnitude  $|B_{(3)}|^2$  at  $r=1$ . On the other hand, the analogous expressions for the potential  $B_{(3)}$  for the two harmonic 4-forms (58) and (60), which are similarly expressible as functions of  $r$  times the three 3-form structures in (62), turn out to have a diverging magnitude at  $r=3$ . In all three cases the  $r$ -dependent prefactors tend to constants at infinity.

Our results on harmonic forms are summarised in the following proposition.

**Proposition 5.1.** *The metric  $\mathbb{A}_8$  of Proposition 3.2 has Spin(7) holonomy and admits an  $L^2$  harmonic 4-form, given by (53) and (56), whose duality is opposite to that of the Cayley form. The metric  $\mathbb{B}_8$  of Proposition 3.3 has Spin(7) holonomy and admits both an anti-self-dual  $L^2$  harmonic 4-form, given by (53) and (58), and a self-dual  $L^2$  harmonic 4-form, given by (53) and (60).*

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